

# Average operators on rectangular Herz spaces\*

Carolina Espinoza-Villalva      Martha Guzmán-Partida

## Abstract

We introduce a family of Herz type spaces considering rectangles instead of balls and we study continuity properties of some average operators acting on them.

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## 1 Introduction

Herz spaces have been studied for many years. The roots of this subject lie on the pioneering work of N. Wiener [11], A. Beurling [2] and C. Herz [9]. Later, these spaces were generalized by other mathematicians in order to study continuity properties of classical operators in harmonic analysis, as well as to develop local versions of Hardy spaces and bounded mean oscillation spaces.

There are several definitions of Herz space. The following is classical and corresponds to the inhomogeneous setting: a measurable function  $f$  belongs to the Herz space  $K_{p,q}^\alpha(\mathbb{R}^n)$ ,  $1 \leq p, q < \infty$ ,  $\alpha \in \mathbb{R}$  if

$$\|f\|_{K_{p,q}^\alpha} := \left( \sum_{k=0}^{\infty} 2^{nk\alpha q} \|f\chi_{C_k}\|_p^q \right)^{1/q} < \infty, \quad (1)$$

and for  $q = \infty$

$$\|f\|_{K_{p,\infty}^\alpha} := \sup_{k \geq 0} \left( 2^{nk\alpha} \|f\chi_{C_k}\|_p \right) < \infty. \quad (2)$$

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Here  $C_0$  is the open unit ball  $B_1(0)$  and  $C_k = B_{2^k}(0) \setminus B_{2^{k-1}}(0)$ ,  $k \in \mathbb{N}$ .

Setting  $\alpha = -1/p$  in (2) we obtain the space  $B^p(\mathbb{R}^n)$  that also can be characterized by mean of the condition ([5], [7])

$$\sup_{R \geq 1} \left( \frac{1}{|B_R(0)|} \int_{B_R(0)} |f(x)|^p dx \right)^{1/p} < \infty \quad (3)$$

and the quantity on the left hand side of (3) defines an equivalent norm to  $\|f\|_{K_{p,\infty}^{-1/p}}$  that is usually denoted by  $\|f\|_{B^p}$ . With any of these norms  $B^p(\mathbb{R}^n)$  turns out to be a Banach space. Moreover, for  $1 \leq p_1 < p_2 < \infty$  we have the inclusions  $B^{p_2}(\mathbb{R}^n) \subset B^{p_1}(\mathbb{R}^n)$  and  $L^\infty(\mathbb{R}^n) \subset B^p(\mathbb{R}^n)$  for every  $p$ .

In this work we will restrict to the context of the space  $B^p(\mathbb{R}^n)$  for  $1 \leq p < \infty$ . Our aim is to explore what happens when we consider rectangles with sides parallel to the coordinate axes instead of balls in (3). As we will see below, although we obtain a smaller space than  $B^p(\mathbb{R}^n)$ , it is still appropriate to study continuity properties of some classical operators. In the context of the present paper, we study continuity properties of some discrete and continuous versions of the classical Hardy average operator. This operator has been extensively studied for many authors on different function spaces. We restrict ourself to consider the most simple versions of this operator in order to make easy the reading of the present paper.

The manuscript is organized as follows: the second section is devoted to introduce the rectangular Herz spaces and to give some examples. In the third section we introduce the average operators to be considered and we prove the continuity of these averages on our spaces.

We will employ standard notation along this work and we will also adopt the convention to denote by  $C$  a constant that could be changing line by line.

## 2 Rectangular Herz spaces

For  $1 \leq p < \infty$ , we define the following space

$$\mathcal{B}^p(\mathbb{R}^n) = \{f \in L_{loc}^p(\mathbb{R}^n) : \|f\|_{\mathcal{B}^p} < \infty\},$$

where

$$\|f\|_{\mathcal{B}^p} := \sup_{\substack{R_j \geq 1 \\ j=1,\dots,n}} \left[ \frac{1}{R_1 \dots R_n} \int_{[-R_1, R_1] \times \dots \times [-R_n, R_n]} |f(x)|^p dx \right]^{1/p}. \quad (4)$$

If the context does not cause confusion, we will simply write  $\mathcal{B}^p$ . Notice that for  $n = 1$ , the spaces  $\mathcal{B}^p(\mathbb{R})$  and  $B^p(\mathbb{R})$  coincide.

Standard arguments (see [1], for example) allow us to see that  $(\mathcal{B}^p, \|\cdot\|_{\mathcal{B}^p})$  is a Banach space. Moreover, it is clear that  $\mathcal{B}^p \subset B^p$  and  $\|\cdot\|_{B^p} \leq \|\cdot\|_{\mathcal{B}^p}$  since Lebesgue measure of balls and cubes are comparable.

**Proposition 1** *The space  $\mathcal{B}^p(\mathbb{R}^n)$  is properly contained in  $B^p(\mathbb{R}^n)$  when  $n \geq 2$ .*

**Proof.** For the sake of clarity, we will consider the case  $n = 2$ .

Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be the function defined as follows:

$$f(x) = \begin{cases} 0 & \text{if } x \notin ([-1, 1] \times \mathbb{R}) \cup (\mathbb{R} \times [-1, 1]), \\ 1 & \text{if } x \in [-1, 1] \times [-1, 1], \\ 2^{1/p} & \text{if } x \in ([-1, 1] \times (1, 2]) \\ & \cup ([-1, 1] \times [-2, -1)) \\ & \cup ((1, 2] \times [-1, 1]) \\ & \cup ([-2, -1] \times [-1, 1]), \\ \cdot & \\ \cdot & \\ \cdot & \\ n^{1/p} & \text{if } x \in ([-1, 1] \times (n-1, n]) \\ & \cup ([-1, 1] \times [-n, -n+1)) \\ & \cup ((n-1, n] \times [-1, 1]) \\ & \cup ([-n, -n+1] \times [-1, 1]), \quad n \geq 2. \end{cases}$$

Take  $R \geq 1$ . We can find  $k \in \mathbb{N}$  such that  $k \leq R < k+1$  and thus

$$\begin{aligned} \frac{1}{|[-R, R]^2|} \int_{[-R, R]^2} |f(x)|^p dx &\leq \frac{1}{4k^2} \int_{[-(k+1), k+1]^2} |f(x)|^p dx \\ &= \frac{1}{4k^2} [1 \cdot 2^2 + 2 \cdot 2^3 + 3 \cdot 2^3 + \dots + (k+1) \cdot 2^3] \\ &\leq \frac{2}{k^2} [1 + 2 + \dots + (k+1)] \\ &= \frac{(k+1)(k+2)}{k^2} \leq 6 \end{aligned}$$

which shows that  $f \in B^p(\mathbb{R}^2)$ . However, if now we consider rectangles of the form  $[-1, 1] \times [-L, L]$  for  $L \geq 2$ , we can pick  $m \in \mathbb{N}$  such that  $m \leq L < m+1$

and therefore

$$\begin{aligned}
\frac{1}{|[-1, 1] \times [-L, L]|} \int_{[-1, 1] \times [-L, L]} |f(x)|^p dx &= \frac{1}{4L} \int_{[-1, 1] \times [-L, L]} |f(x)|^p dx \\
&\geq \frac{1}{4(m+1)} \int_{[-1, 1] \times [-m, m]} |f(x)|^p dx \\
&= \frac{1}{4(m+1)} [1 \cdot 2^2 + 2 \cdot 2^2 + \dots + m \cdot 2^2] \\
&= m/2 \rightarrow \infty \text{ if } m \rightarrow \infty,
\end{aligned}$$

that is,  $f \notin \mathcal{B}^p(\mathbb{R}^2)$ . ■

Using the idea of the previous example we can get a characterization of the space  $\mathcal{B}^p(\mathbb{R}^n)$ . To this end, consider the following subsets of  $\mathbb{R}^n$ :

$$C_{j_1, j_2, \dots, j_n} = C_{j_1} \times C_{j_2} \times \dots \times C_{j_n}$$

where

$$C_0 = [-1, 1] \text{ and } C_j = \{x \in \mathbb{R} : 2^{j-1} < |x| \leq 2^j\}$$

for  $j \in \mathbb{N}$ .

For  $1 \leq p < \infty$  and  $f \in L_{loc}^p(\mathbb{R}^n)$  define

$$\|f\|_{\mathcal{B}^p}^* := \sup_{\substack{j_i \geq 0 \\ i=1, 2, \dots, n}} 2^{-\frac{(j_1 + j_2 + \dots + j_n)}{p}} \|f \chi_{C_{j_1, j_2, \dots, j_n}}\|_p.$$

Now, we can state the following characterization.

**Proposition 2**  $f \in \mathcal{B}^p(\mathbb{R}^n)$  if and only if  $\|f\|_{\mathcal{B}^p}^* < \infty$ . Moreover,  $\|f\|_{\mathcal{B}^p}$  and  $\|f\|_{\mathcal{B}^p}^*$  are equivalent norms.

**Proof.** Assume that  $\|f\|_{\mathcal{B}^p}^* < \infty$ . For  $i = 1, \dots, n$  let  $R_i > 1$  and choose  $j_i \in \mathbb{N}$  such that

$$2^{j_i-1} < R_i \leq 2^{j_i}.$$

We have that

$$\begin{aligned}
\int_{\prod_{i=1}^n [-R_i, R_i]} |f(x)|^p dx &\leq \sum_{k_1=0}^{j_1} \sum_{k_2=0}^{j_2} \dots \sum_{k_n=0}^{j_n} \int_{C_{k_1, k_2, \dots, k_n}} |f(x)|^p dx \\
&\leq \sum_{k_1=0}^{j_1} \sum_{k_2=0}^{j_2} \dots \sum_{k_n=0}^{j_n} 2^{k_1 + k_2 + \dots + k_n} (\|f\|_{\mathcal{B}^p}^*)^p \\
&\leq C 2^{j_1 + j_2 + \dots + j_n} (\|f\|_{\mathcal{B}^p}^*)^p \\
&\leq C R_1 R_2 \dots R_n (\|f\|_{\mathcal{B}^p}^*)^p.
\end{aligned}$$

Hence  $f \in \mathcal{B}^p(\mathbb{R}^n)$  and  $\|f\|_{\mathcal{B}^p} \leq C \|f\|_{\mathcal{B}^p}^*$ .

Conversely, if  $f \in \mathcal{B}^p(\mathbb{R}^n)$ ,  $i = 1, \dots, n$  and  $j_i \geq 0$

$$\begin{aligned} \|f \chi_{C_{j_1, j_2, \dots, j_n}}\|_p^p &= \int_{\prod_{i=1}^n [-2^{j_i}, 2^{j_i}]} |f(x)|^p dx \\ &\leq C \|f\|_{\mathcal{B}^p}^p 2^{j_1 + j_2 + \dots + j_n} \end{aligned}$$

which implies that

$$\|f\|_{\mathcal{B}^p}^* = \sup_{\substack{j_i \geq 0 \\ i=1, 2, \dots, n}} 2^{-\frac{(j_1 + j_2 + \dots + j_n)}{p}} \|f \chi_{C_{j_1, j_2, \dots, j_n}}\|_p \leq C \|f\|_{\mathcal{B}^p}.$$

This concludes the proof. ■

### 3 Continuity of average operators

Average integral operators were considered by Hardy, Littlewood and Pólya in [8]. They proved the following classical inequality:

$$\int_0^\infty \left( \frac{F(x)}{x} \right)^p dx \leq \left( \frac{p}{p-1} \right)^p \int_0^1 f^p(x) dx,$$

where  $1 < p < \infty$ ,  $F(x) = \int_0^x f(t) dt$ ,  $f \geq 0$  and the constant  $\left( \frac{p}{p-1} \right)^p$  is the best possible.

Closely related to this operator is the operator  $H_\varphi$  introduced by Carton-Lebrun and Fosset in [3] and by Xiao in [10] which is pointwisely defined as follows:

$$H_\varphi f(x) := \int_0^1 f(tx) \varphi(t) dt. \quad (5)$$

Xiao in [10] proved continuity of  $H_\varphi$  under appropriate conditions on  $\varphi$  on  $L^p(\mathbb{R}^n)$  and  $BMO(\mathbb{R}^n)$  for  $1 \leq p \leq \infty$ . It is our goal to prove continuity of this and other related operators in our rectangular Herz spaces.

We will start by considering the following discrete version of (5).

Let  $\{r_k\}_{k=1}^\infty$  be a sequence in  $(0, 1]$  which is strictly decreasing and  $\lim_{k \rightarrow \infty} r_k = 0$ . If  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is a Lebesgue measurable function and  $\varphi : \{r_k : k \in \mathbb{N}\} \rightarrow (0, \infty)$  is any function, consider the operator  $H_\varphi^d$  formally defined as

$$H_\varphi^d f(x) = \sum_{k=1}^\infty \varphi(r_k) f(r_k x).$$

Now, notice that a necessary and sufficient condition for the existence of  $H_\varphi^d$  as a bounded operator on  $L^p(\mathbb{R}^n)$  is that

$$\sum_{k=1}^{\infty} r_k^{-n/p} \varphi(r_k) < \infty. \quad (6)$$

Indeed, assuming the convergence of the series in (6), given  $f \in L^p(\mathbb{R}^n)$ ,  $1 \leq p < \infty$ , and using Minkowski inequality we obtain

$$\begin{aligned} \|H_\varphi^d f\|_p &\leq \sum_{k=1}^{\infty} \varphi(r_k) \left( \int_{\mathbb{R}^n} |f(r_k x)|^p dx \right)^{1/p} \\ &= \|f\|_p \sum_{k=1}^{\infty} r_k^{-n/p} \varphi(r_k), \end{aligned}$$

which implies that  $\|H_\varphi^d\|_{L^p \rightarrow L^p} \leq \sum_{k=1}^{\infty} r_k^{-n/p} \varphi(r_k)$ .

Conversely, if  $H_\varphi^d$  is bounded on  $L^p(\mathbb{R}^n)$ , we can consider as Xiao in [10] the function

$$f_\varepsilon(x) = |x|^{-\frac{n}{p}-\varepsilon} \chi_{\{|x|>1\}},$$

where  $0 < \varepsilon < 1$ . It turns out that  $\|f_\varepsilon\|_p = \frac{C_n}{p\varepsilon}$ ,  $C_n$  an  $n$ -dimensional constant and

$$H_\varphi^d f_\varepsilon(x) = \left( \sum_{k=1}^{\infty} r_k^{-\frac{n}{p}-\varepsilon} \varphi(r_k) \right) |x|^{-\frac{n}{p}-\varepsilon} \chi_{\{|x|>1\}}.$$

Thus, same procedure as done in [10] shows that

$$\|H_\varphi^d\|_{L^p \rightarrow L^p}^p \|f_\varepsilon\|_p^p \geq \left[ \varepsilon^\varepsilon \sum_{k=1}^{\infty} r_k^{-\frac{n}{p}-\varepsilon} \varphi(r_k) \right]^p \|f_\varepsilon\|_p^p$$

and therefore

$$\|H_\varphi^d\|_{L^p \rightarrow L^p} \geq \left[ \varepsilon^\varepsilon \sum_{k=1}^{\infty} r_k^{-\frac{n}{p}-\varepsilon} \varphi(r_k) \right] \geq \varepsilon^\varepsilon \sum_{k=1}^{\infty} r_k^{-\frac{n}{p}} \varphi(r_k)$$

for any  $0 < \varepsilon < 1$ . Now, letting  $\varepsilon \rightarrow 0$  we obtain

$$\|H_\varphi^d\|_{L^p \rightarrow L^p} \geq \sum_{k=1}^{\infty} r_k^{-\frac{n}{p}} \varphi(r_k).$$

We have proved the following result.

**Theorem 3** *The operator  $H_\varphi^d$  is a bounded operator on  $L^p(\mathbb{R}^n)$ ,  $1 \leq p < \infty$ , if and only if  $\sum_{k=1}^{\infty} r_k^{-\frac{n}{p}} \varphi(r_k) < \infty$ . In such case*

$$\|H_\varphi^d\|_{L^p \rightarrow L^p} = \sum_{k=1}^{\infty} r_k^{-\frac{n}{p}} \varphi(r_k).$$

We can also consider the following generalization of the operator  $H_\varphi^d$ .

Let  $\Phi : \left\{ r_{k_1}^{(1)} : k_1 \in \mathbb{N} \right\} \times \dots \times \left\{ r_{k_n}^{(n)} : k_n \in \mathbb{N} \right\} \longrightarrow (0, \infty)$  any function, where for every  $j = 1, \dots, n$ , the sequence  $\left\{ r_{k_j}^{(j)} \right\}_{k_j=1}^{\infty} \subset (0, 1]$ , is strictly decreasing and  $\lim_{k_j \rightarrow \infty} r_{k_j}^{(j)} = 0$ . For a Lebesgue measurable function  $f : \mathbb{R}^n \longrightarrow \mathbb{R}$  define formally

$$\mathbb{H}_\Phi^d f(x) = \sum_{k_1=1}^{\infty} \dots \sum_{k_n=1}^{\infty} \Phi\left(r_{k_1}^{(1)}, \dots, r_{k_n}^{(n)}\right) f\left(r_{k_1}^{(1)} x_1, \dots, r_{k_n}^{(n)} x_n\right). \quad (7)$$

With the same proof as in Theorem 3 we can show:

**Theorem 4** *The operator  $\mathbb{H}_\Phi^d$  is a bounded operator on  $L^p(\mathbb{R}^n)$ ,  $1 \leq p < \infty$ , if and only if*

$$\sum_{k_1=1}^{\infty} \dots \sum_{k_n=1}^{\infty} \Phi\left(r_{k_1}^{(1)}, \dots, r_{k_n}^{(n)}\right) \left(r_{k_1}^{(1)}\right)^{-1/p} \dots \left(r_{k_n}^{(n)}\right)^{-1/p} < \infty.$$

*In such case*

$$\|\mathbb{H}_\Phi^d\|_{L^p \rightarrow L^p} = \sum_{k_1=1}^{\infty} \dots \sum_{k_n=1}^{\infty} \Phi\left(r_{k_1}^{(1)}, \dots, r_{k_n}^{(n)}\right) \left(r_{k_1}^{(1)}\right)^{-1/p} \dots \left(r_{k_n}^{(n)}\right)^{-1/p}.$$

Now we will study the action of the operator  $\mathbb{H}_\Phi^d$  on our rectangular Herz spaces defined in the previous section.

For these spaces is even easier the proof of the continuity of the operator  $\mathbb{H}_\Phi^d$ . We provide it for the sake of completeness.

**Theorem 5** *The operator  $\mathbb{H}_\Phi^d$  is a bounded operator on  $\mathcal{B}^p(\mathbb{R}^n)$ ,  $1 \leq p < \infty$ , if and only if*

$$\sum_{k_1=1}^{\infty} \dots \sum_{k_n=1}^{\infty} \Phi\left(r_{k_1}^{(1)}, \dots, r_{k_n}^{(n)}\right) < \infty. \quad (8)$$

In such case

$$\|\mathbb{H}_\Phi^d\|_{\mathcal{B}^p \rightarrow \mathcal{B}^p} = \sum_{k_1=1}^{\infty} \dots \sum_{k_n=1}^{\infty} \Phi \left( r_{k_1}^{(1)}, \dots, r_{k_n}^{(n)} \right).$$

**Proof.** Assuming condition (8), taking  $R_j > 1$ ,  $j = 1, \dots, n$ , and using Minkowski inequality we can see that

$$\begin{aligned} & \left[ \frac{1}{R_1 \dots R_n} \int_{[-R_1, R_1] \times \dots \times [-R_n, R_n]} |\mathbb{H}_\Phi^d f(x)|^p dx \right]^{1/p} \\ & \leq \sum_{k_1=1}^{\infty} \dots \sum_{k_n=1}^{\infty} \Phi \left( r_{k_1}^{(1)}, \dots, r_{k_n}^{(n)} \right) \left[ \frac{1}{R_1 \dots R_n} \int_{[-R_1, R_1] \times \dots \times [-R_n, R_n]} \left| f \left( r_{k_1}^{(1)} x_1, \dots, r_{k_n}^{(n)} x_n \right) \right|^p dx \right]^{1/p} \\ & \leq \sum_{k_1=1}^{\infty} \dots \sum_{k_n=1}^{\infty} \Phi \left( r_{k_1}^{(1)}, \dots, r_{k_n}^{(n)} \right) \|f\|_{\mathcal{B}^p}, \end{aligned}$$

$$\text{and hence } \|\mathbb{H}_\Phi^d\|_{\mathcal{B}^p} \leq \sum_{k_1=1}^{\infty} \dots \sum_{k_n=1}^{\infty} \Phi \left( r_{k_1}^{(1)}, \dots, r_{k_n}^{(n)} \right).$$

Now, if the operator  $\mathbb{H}_\Phi^d$  is bounded on  $\mathcal{B}^p(\mathbb{R}^n)$ , it is enough to consider the function  $f_0 \equiv 1$  because in such case we easily obtain the required reverse inequality. ■

Our next goal is to generalize the operator given by (7). Before to do this, we will define another class of rectangular spaces closely related to  $\mathcal{B}^p$ .

**Definition 6** For  $1 \leq p < \infty$  we define

$$\mathcal{CMO}^p(\mathbb{R}^n) = \{f \in L_{loc}^p(\mathbb{R}^n) : \|f\|_{\mathcal{CMO}^p} < \infty\},$$

where

$$\|f\|_{\mathcal{CMO}^p} := \sup_{\substack{R_j \geq 1 \\ j=1, \dots, n}} \left[ \frac{1}{R_1 \dots R_n} \int_{[-R_1, R_1] \times \dots \times [-R_n, R_n]} |f(x) - f_{R_1 \dots R_n}|^p dx \right]^{1/p}, \quad (9)$$

and  $f_{R_1 \dots R_n}$  is the average of  $f$  on  $[-R_1, R_1] \times \dots \times [-R_n, R_n]$ .

It is not difficult to show that  $(\mathcal{CMO}^p, \|\cdot\|_{\mathcal{CMO}^p})$  is a Banach space if we identify functions that differ by a constant almost everywhere on  $\mathbb{R}^n$ . Also, we obtain an equivalent norm to  $\|\cdot\|_{\mathcal{CMO}^p}$  if we consider the quantity

$$\|f\|_{\mathcal{CMO}^p}^* := \sup_{\substack{R_j \geq 1 \\ j=1, \dots, n}} \inf_{a \in \mathbb{R}} \left[ \frac{1}{R_1 \dots R_n} \int_{[-R_1, R_1] \times \dots \times [-R_n, R_n]} |f(x) - a|^p dx \right]^{1/p}.$$



This space is the rectangular version of the space  $CMO^p$  ([4],[7]) whose elements satisfy the condition

$$\sup_{R \geq 1} \left[ \frac{1}{|Q(0, R)|} \int_{Q(0, R)} |f(x) - f_{Q(0, R)}|^p dx \right]^{1/p} < \infty.$$

Here,  $Q(0, R)$  denotes the cube centered at 0 and side length equal to  $R$ . Clearly,  $\mathcal{B}^p \subset \mathcal{CMO}^p \subset CMO^p$ .

Now, we consider the following operator:

For Lebesgue measurable functions  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ , and  $\phi : [0, 1]^n \rightarrow (0, \infty)$ , we define

$$\mathbb{H}_\phi f(x) := \int_{[0, 1]^n} f(t_1 x_1, \dots, t_n x_n) \phi(t_1, \dots, t_n) dt_1 \dots dt_n. \quad (10)$$

Observe that same proof as given by Xiao in [10], shows that  $\mathbb{H}_\phi$  is a bounded operator on  $L^p(\mathbb{R}^n)$ ,  $1 \leq p < \infty$ , if and only if

$$\int_{[0, 1]^n} t_1^{-1/p} \dots t_n^{-1/p} \phi(t_1, \dots, t_n) dt_1 \dots dt_n < \infty.$$

We will give equivalent conditions for the boundedness of the operator  $\mathbb{H}_\phi$  on the spaces  $\mathcal{B}^p$  and  $\mathcal{CMO}^p$ .

**Theorem 7** *The operator  $\mathbb{H}_\phi$  is a bounded operator on  $\mathcal{B}^p(\mathbb{R}^n)$  and  $\mathcal{CMO}^p(\mathbb{R}^n)$ ,  $1 \leq p < \infty$ , if and only if*

$$\int_{[0, 1]^n} \phi(t_1, \dots, t_n) dt_1 \dots dt_n < \infty.$$

Moreover

$$\|\mathbb{H}_\phi\|_{\mathcal{B}^p \rightarrow \mathcal{B}^p} = \|\mathbb{H}_\phi\|_{\mathcal{CMO}^p \rightarrow \mathcal{CMO}^p} = \int_{[0, 1]^n} \phi(t_1, \dots, t_n) dt_1 \dots dt_n. \quad (11)$$

**Proof.** Just for illustration we prove the equivalence for the space  $\mathcal{CMO}^p(\mathbb{R}^n)$ .

Suppose that the integral in (11) is finite. Then, for  $R_j > 1$ ,  $j = 1, \dots, n$  and  $f \in \mathcal{CMO}^p(\mathbb{R}^n)$  we can easily see that

$$(\mathbb{H}_\phi f)_{R_1 \dots R_n} = \int_{[0, 1]^n} f_{t_1 R_1 \dots t_n R_n} \phi(t_1, \dots, t_n) dt_1 \dots dt_n.$$

Now, by Minkowski inequality and an appropriate change of variable we have that

$$\begin{aligned}
& \left[ \frac{1}{R_1 \dots R_n} \int_{[-R_1, R_1] \times \dots \times [-R_n, R_n]} |\mathbb{H}_\phi f(x) - (\mathbb{H}_\phi f)_{R_1 \dots R_n}|^p dx \right]^{1/p} \\
& \leq \int_{[0,1]^n} \left( \frac{1}{R_1 \dots R_n} \int_{[-R_1, R_1] \times \dots \times [-R_n, R_n]} |f(t_1 x_1, \dots, t_n x_n) - f_{t_1 R_1 \dots t_n R_n}|^p dx \right)^{1/p} \\
& \quad \times \phi(t_1, \dots, t_n) dt_1 \dots dt_n \\
& \leq \|f\|_{\mathcal{CMO}^p} \int_{[0,1]^n} \phi(t_1, \dots, t_n) dt_1 \dots dt_n,
\end{aligned}$$

which implies that

$$\|\mathbb{H}_\phi\|_{\mathcal{CMO}^p \rightarrow \mathcal{CMO}^p} \leq \int_{[0,1]^n} \phi(t_1, \dots, t_n) dt_1 \dots dt_n.$$

For the converse, it suffices to consider the function  $f_0(x) \equiv 1$ . ■

Finally, it should be remarked that Theorems 5 and 7 remain true if we consider homogeneous versions of the spaces  $\mathcal{B}^p$  and  $\mathcal{CMO}^p$ , that is, those defined by taking  $R_j > 0$  for every  $j = 1, \dots, n$  in (4) and (9).

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Departamento de Matemáticas  
 Universidad de Sonora  
 Rosales y Luis Encinas  
 Hermosillo, Sonora, 83000, México  
 Email: carolina.espinoza@mat.uson.mx  
 martha@mat.uson.mx<sup>1</sup>

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<sup>1</sup>Corresponding author